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Study of Relativistic Corrections due to Spin–Orbit Interaction of Fermion–Dyon System Under SU(2) Gauge Potential in Moduli Space

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Abstract Investigating energy eigen values of fermion–dyon system under the fundamental representation of SU(2) isospin group, it is shown that the equation contains an extra term, which depicts interaction of spin moment with a vector field. The study of fermion–dyon interaction in non-abelian gauge theory has also been undertaken and the splitting of energy level of the system has been carried out.

Keywords Vacuum expectation value · Self-duality · Dyon · Non-abelian gauge theory · Spin–orbit interaction · Perturbation · Moduli space · Relativistic approximation · Spin and orbital angular momentum operator · Dyon harmonics

1 Introduction

Monopoles and dyons [1–5] have featured continuously in the variety of context in mathematical and theoretical physics. The theories of pure abelian monopoles [6–14] suffered from many paradoxes some of which could be resolved by taking electric and magnetic charges on the same particle known as dyon [15–21]. Subsequently it became clear that the monopole and dyon can be understood better in non-abelian gauge theories [1, 2, 16–18] and that the reasons for not seeing these particles so far with certainty lie elsewhere rather than due to their inconsistencies in relativistic quantum field theory. Now it is widely recognized that SU(5) grand unified model [19–21] is a gauge theory that contains monopole and dyon solutions and consistently monopoles and dyons have become intrinsic part of all current grand unified theories [22, 23].

BPS monopoles have attracted much attention because they provide a fully three dimensional example of topological solution of Bogomol'nyi type which are static finite energy

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solutions of classical field equations and stable because their energy attains lower topological bound [13, 14, 24], called Bogomol'nyi bound. Much progress in understanding the dynamics of such solutions have been made over the past decades using the idea of moduli space approximation.

Since, it is already proved that under some mathematical conditions of moduli, monopole [25–29] can be converted into a dyon. In order to study the quantal properties of monopoles and dyons for their possible experimental observations, it became necessary to investigate the bound state of dyon with fermion (spin-$1/2$ particles). Undertaking the study of monopoles and dyons in moduli space we have analyzed [25, 27] the extended structure of a non-abelian dyon in presence of a fermion in moduli space. Extending this work in the present paper, behaviour of a fermion in the field of non-abelian dyon has been analyzed and splitting in the energy levels of the system has been studied.

2 Splitting of Energy Levels of Fermion Moving in the Generalized Electromagnetic Field of Non-abelian Dyon in Moduli Space

In order to study the energy eigen value of the system of a fermion moving in the field of non-abelian dyon [10–12, 27–30] incorporating non-relativistic framework with the inclusion of spin effect in moduli space, let us start with the following Schrodinger equation

$$\left[-\frac{1}{2M_0}\hat{\nabla}^2 + F(r)\hat{L}\otimes\sigma_i + l_4\times\phi(r)\right]\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = E\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{2.1}$$

where $\hat{\nabla}$ is $(\vec{P}_i + e_1 g\vec{V}_i)$, $\phi(r)$ is the Higgs potential and the term $F(r)\hat{L}\otimes\sigma_i$ is spin–orbit interaction which will be treated as small perturbation.

Higgs potential can be written in a different form by using a specific kind of gauge transformation which converts the operator $\vec{r}.\,\vec{T}$ to a new operator $\vec{T}_3$ as

$$\phi(r) = -\frac{e_1 e}{r} + \frac{(e_1 g)^2}{2M_0 r^2}, \tag{2.2}$$

where e_1 is the charge on fermion (spin-$1/2$ particle) which is spinning in the field of non-abelian dyon carrying a charge $q = e - ig$.

Though the non-relativistic equation (2.2) is not sufficient to yield precise values for fine structure of energy levels of this system as they are also affected by the relativistic corrections to the kinetic energy operator $-\frac{1}{2M_0}\hat{\nabla}^2$ as much by specific spin–orbit interaction, it can be regarded as a useful guide to an understanding of the role of spin in the bound states of a spin-$1/2$ particle and a non-abelian dyon.

The unperturbed Hamiltonian

$$\hat{H}_0 = \left(-\frac{1}{2M_0}\right)\hat{\nabla}^2 + l_4\times\phi(r), \tag{2.3}$$

represents the familiar central force problem for the system of a fermion moving in the generalized electromagnetic field of non-abelian dyon.

The spin–orbit interaction energy H' for the system is given by

$$\hat{H}' = \left(\frac{e_1 e}{2M_0^2 c^2}\right)\left\langle\frac{1}{r^3}\right\rangle l_2\otimes\hat{S}\otimes\hat{L} - \left[\frac{(e_1 g)^2}{2M_0^3 c^2}\right]\left\langle\frac{1}{r^4}\right\rangle l_2\otimes\hat{S}\otimes\hat{L}. \tag{2.4}$$

To simplify the above equation, we introduce the total angular momentum as

$$\hat{J} \otimes \hat{J} = \hat{L} \otimes \hat{L} + \hat{S} \otimes \hat{S} + 2\hat{L} \otimes \hat{S}.$$

So the Pauli operator for $\hat{H}'$ is given by

$$\begin{aligned}(\hat{H}')_P &= \left(\frac{e_1 e}{4M_0^2 c^2}\right)\left\langle\frac{1}{r^3}\right\rangle l_2 \otimes [(\hat{J} \otimes \hat{J})_P - (\hat{L} \otimes \hat{L})_P - (\hat{S} \otimes \hat{S})_P] \\ &\quad - \left[\frac{(e_1 g)^2}{4M_0^3 c^2}\right]\left\langle\frac{1}{r^4}\right\rangle l_2 \otimes [(\hat{J} \otimes \hat{J})_P - (\hat{L} \otimes \hat{L})_P = (\hat{S} \otimes \hat{S})_P]. \end{aligned} \tag{2.5}$$

Thus the Pauli wave equation becomes

$$(\hat{H})_P \psi_P = [(\hat{H}_0)_P + (\hat{H}')_P]\psi_P = W\psi_P, \tag{2.6}$$

where

$$\begin{aligned}(H_0)_P &= \begin{bmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix}_P \\ &= \begin{bmatrix} -\frac{1}{2M_0}\hat{\nabla}^2 - l_4 \otimes (\frac{e_1 e}{r} + \frac{(e_1 g)^2}{2M_0 r^2}) & 0 \\ 0 & -\frac{1}{2M_0}\hat{\nabla}^2 - l_4 \otimes (\frac{e_1 e}{r} + \frac{(e_1 g)^2}{2M_0 r^2}) \end{bmatrix}_P, \end{aligned} \tag{2.7}$$

and

$$\psi_P = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}_P, \tag{2.8}$$

represents the Pauli wave function. The Pauli wave equation for unperturbed Hamiltonian is given as

$$\begin{bmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix} \begin{bmatrix} \psi_+(0) \\ \psi_-(0) \end{bmatrix} = W(0) \begin{bmatrix} \psi_+(0) \\ \psi_-(0) \end{bmatrix}, \tag{2.9}$$

or

$$\hat{H}_0 \psi_\pm(0) = W(0)\psi_\pm(0). \tag{2.10}$$

This equation can be solved by the method of separation of variables by writing the wave function as

$$\psi = \frac{U(r)}{r} Y_{e_1 g,l,m}(\theta, \phi), \tag{2.11}$$

where $Y_{e_1 g,l,m}(\theta, \phi)$ are dyon harmonics and is independent of r (it may be treated as angular function) and the radial function $\frac{U(r)}{r} = R(r)$ satisfy the equation

$$r^2\left\{\frac{1}{rR(r)}\frac{d^2}{dr^2}(rR) + 2M_0(E - l_4 \otimes \phi)\right\} = -\frac{\Lambda Y_{e_1 g,l,m}(\theta, \phi)}{Y_{e_1 g,l,m}(\theta, \phi)} = l(l+1), \tag{2.12}$$

with

$$\Lambda = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{2.13}$$

whereas ϕ is defined by (2.2).

These equations, on solving give the following energy eigen value

$$E_n = -2M_0 l_4 \otimes (e_1 g)^2[(2n+1) + \{(2n+1)^2 + 4.l_4 \otimes (e_1 g)^2\}^{1/2}]^{-2}, \tag{2.14}$$

where $n = 0, 1, 2, \ldots$ and $\psi_\pm(0)$ will be the wave functions describing the behaviour of a fermion moving in the generalized electromagnetic field of non-abelian dyon and are simply $R_{nl}(r)Y_{e_1 g,l,m}(\theta,\phi)$, where $Y_{e_1 g,l,m}(\theta,\phi)$ may be treated as angular functions. Thus the Pauli wave function for the spin up and the spin down states would be given as

$$\begin{aligned}[\psi_+(0)]_P = \psi_{(n,l,m_l,m_s=+1/2)} &= R_{nl} Y_{e_1 g,l,m}\,|\uparrow\rangle \\ &= \begin{bmatrix} R_{nl} Y_{e_1 g,l,m} \\ 0 \end{bmatrix},\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}[\psi_-(0)]_P = \psi_{(n,l,m_l,m_s=-1/2)} &= R_{nl} Y_{e_1 g,l,m}\,|\downarrow\rangle \\ &= \begin{bmatrix} 0 \\ R_{nl} Y_{e_1 g,l,m} \end{bmatrix}.\end{aligned} \tag{2.16}$$

In the absence of the spin–orbit interaction, both the wave functions corresponds to the same energy. In order to determine the splitting due to spin–orbit interaction, we should choose a representation in which $\hat{H}'$ is diagonal,

$$\begin{aligned}(\phi_1)_P &= \phi_{(n,l,j=j+1/2,m_j)} \\ &= \sqrt{\frac{l+m_j+1/2}{2l+1}}\,\psi_{(n,l,m_l=m_j-1/2,m_s=+1/2)} + \sqrt{\frac{l-m_j+1/2}{2l+1}}\,\psi_{(n,l,m_l=m_j+1/2,m_s=-1/2)},\end{aligned}$$

or

$$(\phi_1)_P = \begin{bmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}}\,R_{nl} Y_{g,l,m_{j-1/2}} \\ \sqrt{\frac{l-m_j+1/2}{2l+1}}\,R_{nl} Y_{g,l,m_{j+1/2}} \end{bmatrix}. \tag{2.17}$$

Then the first order perturbation due to the spin–orbit interaction would be given by

$$\begin{aligned}W_s^{(1)} &= \int d\tau \phi^\dagger (H')_P \phi \\ &= \left(\frac{e_1 e}{4M_0^2 c^2}\right)\int d\tau \left(\frac{1}{r^3}\right)\phi^\dagger l_2 \otimes [(\hat{J}\otimes\hat{J})_P - (\hat{L}\otimes\hat{L})_P - (\hat{S}\otimes\hat{S})_P]\phi \\ &\quad - \left[\frac{(e_1 g)^2}{4M_0^3 c^2}\right]\int d\tau \left(\frac{1}{r^3}\right)\phi^\dagger l_2 \otimes [(\hat{J}\otimes\hat{J})_P - (\hat{L}\otimes\hat{L})_P - (\hat{S}\otimes\hat{S})_P]\phi\end{aligned} \tag{2.18}$$

or

$$\begin{aligned}W_s^{(1)} = \left(\frac{e_1 e}{4M_0^2 c^2}\right)&\left[l_2 \otimes \left\{j(j+1) - l(l+1) - \frac{3}{4}\right\}\right]\int d\tau \left(\frac{1}{r^3}\right) \\ &\times \left[\frac{l \pm m_j + 1/2}{2l+1}|R_{nl}|^2 |Y_{e_1 g,l,m_{j-1/2}}|^2\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{l\mp m_j+1/2}{2l+1}|R_{nl}|^2|Y_{e_1g,l,m_{j+1/2}}|^2\Bigg]\\
&+\left(\frac{(e_1g)^2}{4M_0^2c^2}\right)\left[l_2\otimes\left\{j(j+1)-l(l+1)-\frac{3}{4}\right\}\right]\int d\tau\left(\frac{1}{r^3}\right)\\
&\times\left[\frac{l\pm m_j+1/2}{2l+1}|R_{nl}|^2|Y_{e_1g,l,m_{j-1/2}}|^2\right.\\
&\left.+\frac{l\pm m_j+1/2}{2l+1}|R_{nl}|^2|Y_{e_1g,l,m_{j+1/2}}|^2\right],
\end{aligned}
\tag{2.19}
$$

where the plus and minus sign corresponds to $j=l+1/2$ and $j=l-1/2$ respectively. After integrating we get,

$$
W_s^{(1)}=\begin{cases}+(\frac{e_1e}{4M_0^2c^2})l_2\otimes l\langle\frac{1}{r^3}\rangle+(\frac{(e_1g)^2}{4M_0^3})l_2\otimes l\langle\frac{1}{r^3}\rangle & \text{for } j=l+1/2,\\ -(\frac{e_1e}{4M_0^2c^2})l_2\otimes(l+1)\langle\frac{1}{r^3}\rangle+(\frac{(e_1g)^2}{4M_0^3})l_2\otimes(l+1)\langle\frac{1}{r^3}\rangle & \text{for } j=l-1/2,\end{cases}
\tag{2.20}
$$

where

$$
\left.\begin{array}{r}\langle\frac{1}{r^3}\rangle=\int_0^\infty\frac{1}{r^3}|R_{nl}|^2r^2dr=\frac{1}{n^3l(l+1)(l+1/2)}\frac{1}{a_0^2}\\ \langle\frac{1}{r^4}\rangle=\int_0^\infty\frac{1}{r^4}|R_{nl}|^2r^2dr=\frac{3-5n^3(l+1/2)}{n^5(l-1/2)(l+1/2)(l+3/2)}\frac{1}{a_0^2}\end{array}\right\}.
\tag{2.21}
$$

The splitting of energy levels corresponding to quantum number n is

$$
\begin{aligned}
W&=W^{(0)}+W_s^{(1)}\\
&=\begin{cases}E_n-\frac{E_ne_1e(l_2\otimes l)}{2M_0^2c^2n^3l(l+1)(2l+1)}\frac{1}{a_0^2}-\frac{E_n(e_1g)^2(l_2\otimes l)[3-5n^3(l+1/2)]}{M_0^4c^2n^5(2l+1)(2l-1)(2l+3)a_0^2} & \text{for } j=1+1/2,\\ E_n-\frac{E_ne_1e[l_2\otimes(l+1)]}{2M_0^2c^2n^3l(l+1)(2l+1)}\frac{1}{a_0^2}-\frac{E_n(e_1g)^2[l_2\otimes(l+1)][3-5n^3(l+1/2)]}{M_0^4c^2n^5(2l+1)(2l-1)(2l+3)a_0^2} & \text{for } j=1-1/2,\end{cases}
\end{aligned}
\tag{2.22}
$$

where E_n is given by (2.14) and the Bohr radius for this system is given by

$$
a_0=\left[\frac{(e_1g)^2+1}{M_0e_1e}\right].
\tag{2.23}
$$

Equation (2.22) gives the splitting of energy levels corresponding to quantum number n for $j=l+1/2$ and $j=l-1/2$ respectively.

3 Discussion

Non-relativistic Schrodinger equation for fermion–dyon system is given by (2.1) which is treated under the Higgs potential (2.2). The unperturbed Hamiltonian is given by (2.3) which represents that the system of fermion moving in the generalized electromagnetic field of non-abelian dyon can be handled as central force problem. Spin orbit interaction for the system is given by (2.4) and Pauli wave equations for perturbed and unperturbed Hamiltonian are given by (2.6) and (2.9) respectively. Energy eigen value of the system is given by (2.14) and first order perturbation is given by (2.18), (2.19) and (2.20). Equation (2.22) gives splitting of energy levels whereas (2.23) gives Bohr radius of the system.

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